MATH 303 – Measure Theory
Homework 1

Please upload a pdf of your solutions by 23:59 on Monday, September 29. The assignment will be graded out of 10 points, taking into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. Let $(X, \mathcal{B}, \mu)$ be a measure space. In this problem, you will establish two methods for producing new measures from μ.

(a) Let Y be a set and $f : X \to Y$ a function. Show that $f_*\mathcal{B} = \{E \subseteq Y : f^{-1}(E) \in \mathcal{B}\}$ is a σ-algebra on Y.

(b) Define $f_*\mu : f_*\mathcal{B} \to [0, \infty]$ by $(f_*\mu)(E) = \mu\left(f^{-1}(E)\right)$. Show that $f_*\mu$ is a measure.

(c) Let $A \in \mathcal{B}$. Show that $\mu_A : \mathcal{B} \to [0, \infty]$ defined by $\mu_A(E) = \mu(E \cap A)$ is a measure.

Solution: (a) We need to check that $f_*\mathcal{B}$ satisfies the three σ-algebra axioms (see Definition 2.1). In each step, we use the corresponding property of the σ-algebra $\mathcal{B}$.

First, $f^{-1}(Y) = X \in \mathcal{B}$, so $Y \in f_*\mathcal{B}$.

Second, suppose $E \in f_*\mathcal{B}$. This means $f^{-1}(E) \in \mathcal{B}$, so $X \setminus f^{-1}(E) \in \mathcal{B}$. But

$$f^{-1}(Y \setminus E) = f^{-1}(Y) \setminus f^{-1}(E) = X \setminus f^{-1}(E).$$

Therefore, $Y \setminus E \in f_*\mathcal{B}$.

Third, let $(E_n)_{n\in\mathbb{N}}$ be a sequence of elements of $f_*\mathcal{B}$. Then $f^{-1}(E_n) \in \mathcal{B}$ for each $n \in \mathbb{N}$, so $\bigcup_{n\in\mathbb{N}} f^{-1}(E_n) \in \mathcal{B}$. Noting that

$$f^{-1}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(E_n),$$

we conclude that $\bigcup_{n\in\mathbb{N}} E_n \in f_*\mathcal{B}$.

Thus, $f_*\mathcal{B}$ is a σ-algebra on Y.

(b) By construction, if $E \in f_*\mathcal{B}$, then $f^{-1}(E) \in \mathcal{B}$, so $f_*\mu$ is well-defined on $f_*\mathcal{B}$. Moreover, $(f_*\mu)(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$, since μ is a measure on $\mathcal{B}$. It remains to check that $f_*\mu$ is countably additive. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of $f_*\mathcal{B}$. Then the sets $f^{-1}(E_n)$ are pairwise disjoint elements of $\mathcal{B}$, so applying countable additivity of the measure μ, we have

$$(f_*\mu)\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \mu\left(f^{-1}\left(\bigsqcup_{n\in\mathbb{N}} E_n\right)\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} f^{-1}(E_n)\right) = \sum_{n=1}^{\infty} \mu\left(f^{-1}(E_n)\right) = \sum_{n=1}^{\infty} (f_*\mu)(E_n).$$

This proves that $f_*\mu$ is a measure on $(Y, f_*\mathcal{B})$.

(c) The σ-algebra $\mathcal{B}$ is closed under finite intersections. Hence, for every $E \in \mathcal{B}$, we have $E \cap A \in \mathcal{B}$, so μ_A is well-defined. We now check that μ_A satisfies the properties required for a measure. By definition, $\mu_A(\emptyset) = \mu(\emptyset \cap A) = \mu(\emptyset) = 0$. Given a sequence $(E_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{B}$, the sets $E_n \cap A$ are also pairwise disjoint, so

$$\mu_A\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \mu\left(\left(\bigsqcup_{n\in\mathbb{N}} E_n\right)\cap A\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} (E_n \cap A)\right) = \sum_{n=1}^{\infty} \mu(E_n \cap A) = \sum_{n=1}^{\infty} \mu_A(E_n).$$

Thus, μ_A is a measure.